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PART FIRST OF

AN

ELEMENTARY TREATISE

ON THE

APPLICATION OF

The Algebraic Analysis

TO

GEOMETRY.

BIBLIOTHECA·BODLEIANA

BY W. S. B. WOOLHOUSE.

IN Part 2, denominated "GEOMETRY OF THREE DIMENSIONS," will be given the derivations of the formulæ relating to lines and planes in space, curved surfaces, curves of double curvature, &c.

And Part 3, will comprise numerous miscellaneous problems, elucidating the varied and extensive application of the formulæ and principles laid down in the preceding Parts.

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PREFACE.

THE decidedly great advantage of the Modern Mathematicians over the Ancients, has almost entirely arisen from the introduction and refinement of the Algebraic Analysis, united with the Differential and Integral Calculus; and particularly from the truly elegant and systematic mode which has been adopted in their application to problems connected with Geometry.

It is but recently that the plan of determining the relative geometrical positions of points, by means of their ordinates related to fixed axes, has engaged the attention, generally, of our English writers. By means of this simple and ingenious contrivance, almost an unlimited power has been acquired in the resolution of physical problems, which require the aid of Geometry, and the differental and integral calculus has been employed with surprising efficacy. In many cases, wherein the greatest effort of the imagination would be inadequate to afford any clear or correct reasonings, the relations of the figure and the necessary conditions can be fully and distinctly put down by means of analytical equations; and the several operations, which are generally attended with much facility, depend, in a great measure, on the general principles of elimination.

The following concise introductory treatise was commenced with a view of setting forth, in the most simple and perspicuous manner, the principal formulæ, which may be found useful in the solutions of the various descriptions of geometrical and physical problems; and furthermore to condense them into one small volume, with a progressive arrangement, so as afford the utmost facility in referring to such principles as may be applicable to any particular subject.

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THE

MODERN APPLICATION
OF THE
ALGEBRAIC ANALYSIS
TO
GEOMETRY.

SECTION I.

DEFINITIONS AND FIRST PRINCIPLES.

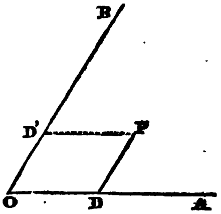
Article 1. *A Function* is any analytical expression, containing one or more variable quantities, combined or not with constant quantities; it is called a function of the variable quantity, or quantities, which it contains. Thus $x^2 + a\,x$, $\sqrt{(a^2 - x^2)}$ are algebraic functions of x; $\tan^{-1} x$, which denotes the length of the circular *arc* whose radius is unity and tangent x, is a trigonometric function of x; and $x^2 + y^2 + xy$, $ax + by - \frac{x^2 y}{c}$ are algebraic functions of x and y.

2. It hence appears that functions of any quantities have their values entirely dependent on the values of those quantities. For when the values of any quantities are given, the values of any functions in which they may be involved become immediately determinable.

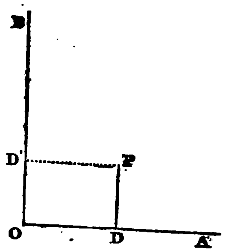
3. A function of any variable or variables is generally denoted by prefixing one of the characters F, f, ϕ, ψ, &c. Sometimes functions of single variables are distinguished by their capitals; thus X, X', X'', &c. being taken to represent functions of x, and Y, Y', Y'', &c. functions of y.

4, *Functions are said to be the same* when the variable quantity, or quantities, which they contain, enter into their respective expressions in exactly the same manner. Thus $x^2 + ax$ is the same function of x that $y^2 + ay$ is of y, and $x^2 + y^2 + xy$ is the same function of x and y that $u^2 + v^2 + uv$ is of u and v. Supposing ϕ to denote the characteristic of $x^2 + ax$, that is, supposing $x^2 + ax$ to be indicated by ϕx, the expression $y^2 + ay$ will thence be indicated by ϕy; Similarly if $x^2 + y^2 + ay$ be represented by $f(x, y)$, the value of $u^2 + v^2 + uv$ will hence be denoted by $f(u, v)$. Also assuming $\psi(x, y)$ for $ax + by - \frac{x^2 y}{c}$, by the same transfer of notation $\psi(u, v)$ will denominate the expression $au + bv - \frac{u^2 v}{c}$.

5. In order to express geometrical positions algebraically points are referred to what are called *the axes of co-ordinates*. These axes of co-ordinates are two fixed indefinite right lines OA, OB, taken on a given plane and parallel to two given lines so as to form a given angle AOB. The point O from whence they proceed is called the origin.



If any point P be assumed in the same plane and the parallelogram $PD'OD$ completed, the portions OD, OD', taken from the origin along the axes, are called *the co-ordinates of the point P* referred to the axes OA and OB. These ordinates are usually denoted by x and y, viz: OD by x and OD' or DP by y; and OA is hence called the *axis of x* and OB the *axis of y*.



6. When the axes $O\,A$, $O\,B$ are perpendicular to each other, they are called *rectangular axes;* and $O\,D$, $D\,P$ are then called the *rectangular co-ordinates* of the point P.

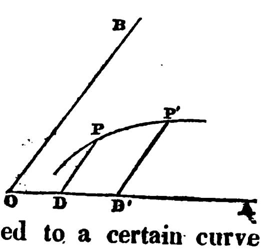
7. If the position of a point be on the contrary side of the axis $O\,A$ its ordinate y will have a *negative value;* and if it be on the contrary side of $O\,B$, or if the point D be on the contrary side of the origin O, we shall similarly have its ordinate x *negative.* For the ordinates are then estimated from the origin in the opposite directions.

8. Particular ordinates which appertain to fixed points are distinguished thus, $x'\ y'$, $x''\ y''$, &c.; and the points thus determined are called the points $x'\ y'$, $x''\ y''$, &c. A point on $O\,A$ has $y' = o$ and is denoted by $x'\ o$, and a point on $O\,B$ is denoted by $y'\ o$.

9. If the ordinates x and y are so related as to fulfil a given equation in which they are involved with constants, we shall have particular values of y for each particular value of x. Thus let

$$f\,(x,\,y) = o$$

be the equation which connects the ordinates x and y, $f\,(x,\,y)$ denoting some given function.



Then if a series of ordinates $O\,D$, $O\,D'$, &c. be assumed as values of x, the ordinates $D\,P$, $D'\,P'$, &c. which are the respectively corresponding values of y, will be determined by the solutions of the proposed equation; and the point P is hence limited to a certain curve line $P\,P'$. This curve line is said to be represented or indicated by the given equation, which is usually called *the equation of the curve.*

10. Rectangular axes are more extensively used, and, in most cases, are more simple in their application than those which are oblique. Sometimes, however, in particular geometrical investigations, the axes may, with great advantage, be taken parallel to certain given lines in the figure; yet, as rectangular axes are generally of more utility, we shall confine ourselves to the consideration of them in the subsequent investigations.

SECTION II.

EQUATIONS OF THE FIRST DEGREE.

11. The equation of the first degree is of the form

$$a x + b y + c = o, \dots. (1).$$

wherein each of the constants a, b, c, may be either *positive* or *negative*.

By solving for y, it gives

$$y = -\frac{a}{b} x - \frac{c}{b};$$

and hence, assuming

$$m = -\frac{a}{b},\ h = -\frac{c}{b},$$

it becomes

$$y = m x + h \dots. (2).$$

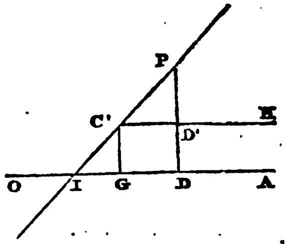
This form of equation, in which the constants m and h may be either *positive* or *negative*, evidently then comprehends in it all cases whatever of the *first degree*, the same as the above equation marked (1).

12. Let C' be any assumed invariable point in the locus determined by the equation

$$y = mx + h,$$

whose ordinates $OG = x'$, $GC' = y'$ will hence satisfy the equation

$$y' = mx' + h.$$



Let also P be any other point in the locus whose ordinates $O\,D$, $D\,P$ fulfil the equation.

$$y = m\,x + h;$$

and, by deducting the above, we derive

$$y - y' = m\,(x - x').$$

But, by drawing $C'D'H'$ parallel to $O\,A$ the axis of x, it is clearly seen that $x - x' = GD = C'D'$ and $y - y' = D'\,P$; therefore

$$D'P = m\,.\,C'D'$$

$$\text{and } \therefore\ m = \frac{D'P}{C'D'} = tan \angle\, PC'H.$$

Since this property applies equally to all points P whose ordinates fulfil the proposed equation, the determined locus is evidently a *straight line* through the point C' making an angle with the axis of x whose tangent $= m$.

Similarly the equation

$$a\,x + b\,y + c = o$$

determines a right line intersecting the axis of x at an angle whose tangent $= -\dfrac{a}{b}$, (see article 11).

13. It hence appears that all equations of the *first degree* determine *straight lines;* also, (11,) that these equations may be reduced to either of the forms

$$a\,x + b\,y + c = o \;..\; (1,)$$

$$y = m\,x + h \;...\; (2.)$$

The latter of these is the more simple in its application, in consequence of its involving only two constants, m, h. However, as equations will not always reduce to this form without fractions, we shall generally deduce the several formulæ for both cases. And it may be observed that formulæ applying to the former may be easily rendered suitable to the latter by substituting

m for a; -1 for b, and h for c,

as the equation would then become

$$mx - y + h = o$$

$$\text{or } y = mx + h,$$

agreeing with the latter equation.

It may also be observed that, *vice versæ*, formulæ belonging to the latter mode of equation may be appropriated to the former by substituting, (11,)

$$-\frac{a}{b} \text{ for } m, \text{ and } -\frac{c}{b} \text{ for } h \ldots.$$

14. Since, (12,) m is the tangent of the angle which the line makes with the axis of x, by calling this angle ω, the equation of the line may be expressed thus:

$$y = x \tan \omega + h,$$

15. When $y = o$, the point P must obviously be situated on the axes of x, and will therefore determine the intersection of the line with $O\ A$. Put $y = o$ in the equation

$$ax + by + c = o$$

and let x'' be the corresponding value of x and we shall thence have

$$x'' = -\frac{c}{a},$$

that is,

$$OI = -\frac{c}{a},$$

I being the intersection of the line with the axis of x. Similarly, by taking $x = o$, we find the intersection on the axis of y to be determined by

$$y'' = -\frac{c}{b}.$$

16. By dividing the equation of the line by c and changing the signs, it becomes

$$-\frac{a}{c}x-\frac{b}{c}y-1=o,$$

which is hence equivalent to

$$\frac{x}{x''}+\frac{y}{y''}-1=o,$$

$$\text{or } \frac{x}{x''}+\frac{y}{y''}=1.$$

This equation may therefore be regarded as determining that straight line which cuts portions from off the axes of x and y, estimated from the origin, respectively equal to x'' and y''.

17. When $c=o$, or $h=o$, the equation becomes of the form

$$ax+by=o$$

$$\text{or } y=mx,$$

which is satisfied with $x=o$, $y=o$.
This shews the line to pass through the origin. In this case we have $x''=o$, $y''=o$.

18. When $a=o$, the equation is

$$by+c=o$$

$$\text{or } y=-\frac{c}{b}, \text{ a constant value.}$$

In this case the value of x is arbitrary; and therefore the line is parallel to the axis of x at the distance $-\frac{c}{b}$ on the side where y is positive.

If $b=o$, or the equation be of the form

$$ax+c=o,$$

we shall have

$$x=-\frac{c}{a} \text{ and } y \text{ arbitrary;}$$

and consequently the line is parallel to the axis of y at the distance $-\frac{c}{a}$ on the side where x is positive.

19. When the equation becomes simply $b\,y = o$, or $y = o$, it evidently determines the axis of x; and similarly when it is $x = o$ it indicates the axis of y.

20. We have seen, (12,) that the tangents of the angles which the straight lines

$$a\,x + b\,y + c = o,$$
$$y = m\,x + h$$

make with the axis of x are respectively $= -\frac{a}{b}$ and m. It therefore appears that equations, which have the values of $\frac{a}{b}$, or of m, the same, determine parallel lines. Thus

$$a\,x + b\,y + c = o,$$
$$a\,x + b\,y + c' = o,$$
$$n\,a\,x + n\,b\,y + c'' = o$$

represent parallel lines.

And similarly, the equations

$$y = m\,x + h,$$
$$y = m\,x + h'$$

determine parallel lines.

SECTION III.

FORMULÆ, &c. RELATING TO STRAIGHT LINES.

21. *To express the equation of a straight line which shall pass through a given point.*

As one condition is here imposed upon the line, one of the constants a, b, c will become a function of the other two and the invariable ordinates x' y' of the given point, since they have to satisfy the condition

$$a\,x' + b\,y' + c = o$$

Hence eliminating c, by taking its value $-(a\,x' + b\,y')$, the equation of the line, $a\,x + b\,y + c = o$, becomes

$$ax + by - (ax' + by') = o,$$
$$\text{or } a(x - x') + b(y - y') = o,$$
which determines a straight line passing through the proposed point $x' y'$.

Or it may evidently be expressed thus:
$$y - y' = m(x - x'),$$
m being the tangent of the inclination of the line with the axis of x.

22. Cor. It hence appears that the equation
$$y - y' = (x - x') \tan \omega$$
determines a straight line passing through the point $x' y'$ and making the angle ω with the axis of x.

23. *To determine the equation of a straight line passing through two given points.*

Let $x' y', x'' y''$ be the co-ordinates of the two points; as these points are situated in the line, the constants m, h, must evidently fulfil the equations
$$y' = m x' + h,$$
$$y'' = m x'' + h;$$
from which we find
$$m = \frac{y'' - y'}{x'' - x'},\ h = \frac{x'' y' - x' y''}{x'' - x'}.$$
As the line passes through both of the points $x' y', x'' y''$ its equation, (21,) is hence
$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x'),$$
$$\text{or } y - y'' = \frac{y'' - y'}{x'' - x'} (x - x''),$$
wherein $x' y', x'' y''$ are the ordinates of the two given points and $x\ y$ any point whatever in the line.

24. *Given the equations of two straight lines to find the co-ordinates of the point where they intersect.*

Let $a\,b\,c, a'\,b'\,c'$ be the constants which belong to their respective equations; and let $x\ y$ be the ordinates of the point of intersection; this point being posited in both lines, its ordinates $x\,y$ must satisfy both of the equations

$$a\,x + b\,y + c = o,$$
$$a'\,x + b'\,y + c' = o.$$

From these we find

$$x = \frac{b\,c' - c\,b'}{a\,b' - b\,a'} \text{ and } y = -\,\frac{a\,c' - c\,a'}{a\,b' - b\,a'},$$

which determine the required point.

If the equations of the lines be of the forms

$$y = m\,x + h,$$
$$y = m'\,x + h'$$

we shall have

$$x = -\,\frac{h - h'}{m - m'},\ y = -\,\frac{m'\,h - m\,h'}{m - m'}.$$

25. *The co-ordinates of two points being given to find an expression for their distance, or the length of the line which joins them.*

Let $x\,y, x'\,y'$ be the given ordinates; then drawing lines from the two points parallel to the axes, viz: one parallel to the axis of x and the other parallel to the axis of y, we shall obviously have a right angled triangle whose legs are $x - x'$, $y - y'$; and thus the square of the required line is found equal to

$$(x - x')^2 + (y - y')^2,$$
$$\text{or } l = \surd\,\{(x - x')^2 + (y - y')^2\}.$$

26. *The equation of a straight line being given to express its inclination with the axis of x.*

The equation of the line being

$$a\,x + b\,y + c = o,$$

its inclination ω with the axis of x is, (12,) determined by

$$tan\ \omega = -\,\frac{a}{b},$$

which gives

$$cos\ \omega = \frac{1}{\surd\,(1 + tan^2\,\omega)} = -\,\frac{b}{\surd\,(a^2 - b^2)}$$

and $\sin \omega = \dfrac{\tan \omega}{\sqrt{(1 + \tan^2 \omega)}} = \dfrac{a}{\sqrt{(a^2 + b^2)}}$.

27. When the equation of the line is of the form

$$y = m x + h,$$

we have, (14,)

$$\tan \omega = m \text{ and } \therefore \cos \omega = \frac{1}{\sqrt{(1 + m^2)}},\ \sin \omega = \frac{m}{\sqrt{(1 + m^2)}}.$$

28. *Cor.* Let x'' determine the intersection of the line with the axis of x, as in article 15, and we shall evidently have, for the perpendicular from the origin on the proposed line, $p = x'' \sin \omega$. Hence, substituting the value of x'', (article 15,) we have

$$p = \frac{-c}{\sqrt{(a^2 + b^2)}}.$$

29. *Given the equations of two right lines to find their angle of inclination.*

Suppose $a\, b\, c, a'\, b'\, c'$ to be the constants contained in their equations and let i denote the required inclination; also let ω, ω' be the inclinations of the two lines with the axis of x. Then, (12,)

$$\tan \omega = -\frac{a}{b},\ \tan \omega' = -\frac{a'}{b'};$$

and we evidently have

$$i = \omega - \omega' \text{ and } \therefore \tan i = \frac{\tan \omega - \tan \omega'}{1 + \tan \omega \tan \omega'}.$$

Therefore by substitution

$$\tan i = \frac{a' b - b' a}{a a' + b b'}.$$

Hence also

$$\cos i = \frac{1}{\sqrt{(1 + \tan^2 i)}} = \frac{a a' + b b'}{\sqrt{\{(a^2 + b^2)(a'^2 + b'^2)\}}},$$

$$\sin i = \cos i \times \tan i = \frac{a' b - b' a}{\sqrt{\{(a^2 + b^2)\, a'^2 + b'^2)\}}}.$$

30. *Cor.* 1. If the lines be parallel, $i = o$ and $\sin i = 0$;

$$\therefore a' b - b' a = o$$

$$\text{or } \frac{a'}{b'} = \frac{a}{b},$$

agreeing with (20.)

31. *Cor.* 2. When the lines are perpendicular, $\cos i = o$ and hence

$$a a' + b b' = o.$$

32. *Cor.* 3. If the equations of the lines be

$$y = m x + h,$$
$$y = m' x + h',$$

we shall have

$$\tan \omega = m, \tan \omega' = m';$$

$$\tan i = \frac{m - m'}{1 + m m'},$$

$$\cos i = \frac{1 + m m'}{\sqrt{\{(1 + m^2)(1 + m'^2)\}}},$$

$$\sin i = \frac{m - m'}{\sqrt{\{(1 + m^2)(1 + m'^2)\}}}.$$

When the lines are parallel, $\sin i = o$,

$$m = m'.$$

And when they are perpendicular, $\cos i = o$, and

$$1 + m m' = o.$$

33. *To determine the equation of a straight line inclined at a given angle with a given straight line.*

In the last proposition, article 29, let $a\, b\, c$ be the constants to the equation of the given line and $a'\, b'\, c'$ those of the one required. Then

$$\tan \omega' = \tan (\omega - i) = -\frac{\tan i - \tan \omega}{1 + \tan \omega \tan i};$$

or, substituting $-\frac{a}{b}, -\frac{a'}{b'}$, for $\tan \omega$, $\tan \omega'$,

$$\frac{a'}{b'} = \frac{a + b \tan i}{b - a \tan i} = \frac{a \cos i + b \sin i}{b \cos i - a \sin i}.$$

Hence the required equation is

$$(a \cos i + b \sin i)\, x + (b \cos i - a \sin i)\, y + c'' = o,$$

wherein c'' is an indeterminate constant whose value may be found from another condition.

34. *Cor.* 1. If the line be required to pass through a given point $x'\,y'$ its equation, (21,) will therefore be

$$(a \cos i + b \sin i)\,(x - x') + (b \cos i - a \sin i)\,(y - y') = 0.$$

35. *Cor.* 2. When the line is perpendicular to the proposed one its equation will be

$$b\,x - a\,y + c'' = o.$$

36. *Cor.* 3. When the line is perpendicular to the given one and also passes through a given point $x'\,y'$ its equation will hence be

$$b\,(x - x') - a\,(y - y') = o.$$

The general equation of a straight line which is perpendicular to one determined by the equation

$$y = m\,x + h$$

$$\text{is } y = -\frac{1}{m}\,x + h';$$

and, when passing through a given point $x'\,y'$ is

$$y - y' = -\frac{1}{m}\,(x - x').$$

37. *To express the length of the perpendicular on a given line from a given point.*

Let $a\,x + b\,y + c = o$ be the equation of the line and $x'\,y'$ the ordinates of the given point. Then, (36,) the equation of the perpendicular is

$$b\,(x - x') - a\,(y - y') = o,$$

from which, together with the above equation of the given line, we find the point where it is intersected by this perpendicular to be determined by ordinates $x\,y$ whose values are

$$x = \frac{b^2\,x' - a\,b\,y' - ac}{a^2 + b^2},\; y = \frac{a^2\,y' - a\,b\,x' - b\,c}{a^2 + b^2}.$$

Now, (25,) the length of the perpendicular $P = \sqrt{\{(x-x')^2+(y-y')^2\}}$.

Hence by substitution

$$x - x' = \frac{-a(ax'+by'+c)}{a^2+b^2}, y - y' = \frac{-b(ax'+by'+c)}{a^2+b^2}$$

$$\text{and } P = -\frac{ax'+by'+c}{\sqrt{(a^2+b^2)}}.$$

If the equation of the line be of the form $y = mx + h$, substituting m for a, -1 for b and h for c, (see 13,) and we shall have

$$P = -\frac{(mx'+h)-y'}{\sqrt{(1+m^2)}}.$$

38. *Cor.* 1. When the given point is the origin, we have $x' = o$, $y' = o$; and hence, for the perpendicular from the origin on the given line, we have

$$P = \frac{-c}{\sqrt{(a^2+b^2)}},$$

$$\text{or } P = \frac{-h}{\sqrt{(1+m^2)}}.$$

39. *Cor.* 2, By (20,) the equations

$$ax+by+c=o,$$
$$ax+by+c'=o$$

represent parallel lines.

Now, (88,) the respective perpendiculars from the origin are

$$p = \frac{-c}{\sqrt{(a^2+b^2)}}, p' = \frac{-c'}{\sqrt{(a^2+b^2)}};$$

the difference of these gives the perpendicular distance of the said lines $=$

$$\frac{c \sim c'}{\sqrt{(a^2+b^2)}}$$

Similarly the perpendicular distance between the parallel lines

$$y = mx + h,$$
$$y = mx + h'$$

is equal to

$$\frac{h \backsim h'}{\surd\,(1+m^2)}.$$

40. *To express the distance between a given point $x'\,y'$ on a given line and its intersection with another given line.*

Let the equation of the given line, which passes through the point $x'y'$, be

$$a'\,(x-x')+b'\,(y-y')=o\,;$$

and that of the other given line

$$a\,x+b\,y+c=o.$$

Let xy be their point of intersection and its ordinates will be the same in both equations. The latter one being put in the form

$$a\,(x-x')+b\,(y-y')+(a\,x'+b\,y'+c)=o,$$

we shall, by means of it and the former, find

$$x-x'=\frac{b'\,(a\,x'+b\,y'+c)}{a'\,b-b'\,a},\; y-y'=\frac{-a'\,(a\,x'+b\,y+c)}{a'\,b-b'\,a}.$$

Therefore, for the required distance,

$$D=\surd\left\{(x-x')^2+(y+y')^2\right\}=\frac{a\,x'+b\,y'+c}{a'\,b-b'\,a}\,\surd\,(a'^2+b'^2).$$

Otherwise,

Let p be the perpendicular from the point $x'\,y'$ on the line $a\,x+b\,y+c=o$, Then, i being the angle of inclination of the lines, we obviously have

$$p=D\,\sin i$$

$$\therefore D=\frac{p}{\sin i}.$$

Hence, substituting the values of p and $\sin i$ already laid down, (37, 29,) we get

$$D=\pm\frac{a\,x'+b\,y'+c}{a'\,b-b'\,a}\,\surd\,(a'^2+b'^2).$$

41. *Cor.* When the equations are

$y=m\,x+h$ for the intersected line

and $y-y'=m'\,(x-x')$ for the line passing through the point $x'y'$, substitute m, m' respectively for a, a' and -1 for b, b', and h for c, (13,) and

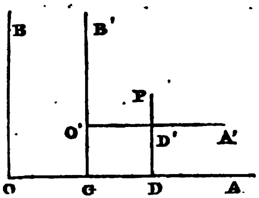
$$D = \frac{m x' + h - y'}{m - m'} \sqrt{(1 + m'^2)}.$$

SECTION IV.

TRANSFORMATION OF THE AXES.

42. With the view of expressing any particular lines or curves, being the loci of points, by algebraic equations, we are manifestly at liberty to assign to the origin and the axes any positions whatever, relative to the said loci; and hence, when the equation of a locus is complex, it becomes sometimes useful to assume another position of the axes which will reduce it to a more simple form. This transformation, which is called *the transformation of co-ordinates*, is effected by expressing the original in terms of the new co-ordinates, of any point, which will, of course, be ready for substitution in any equation or formula, appertaining to the former axes, so as to produce the equivalent involving the new co-ordinates. For these operations the three following propositions are necessary.

43. *An expression involving the two rectangular co-ordinates being given to find the corresponding expression in terms of the co-ordinates when the origin is transferred to a given point, the axes retaining a parallel position.*



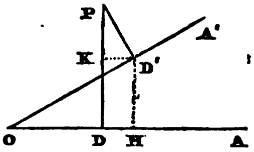
Let O' be the given point to which the origin is to be transferred; and let its position referred to the axes OA, OB be $OG=a$, $GO'=b$; also let the position of the point P related to the new axes $O'A'$, $O'B'$ parallel to OA, OB, be $x'y'$ viz $O'D'=x'$ and $D'P=y'$. Then is

$$OD=x=O'D'+OG=x'+a,$$
$$DP=y=D'P+GO'=y'+b,$$

which substituted for x and y will give the expression required, wherein $a\,b$, the ordinates of the new origin O', will be given constants and $x'\,y'$ the ordinates of P referred to the new axes.

By this means we transfer the origin O to a point whose ordinates are $x=a$, $y=b$.

44. *An expression involving the ordinates of a point referred to two rectangular axes being given to find the corresponding expression when the point is referred to two other rectangular axes making a given angle with the former and proceeding from the same origin.*



We shall omit the axis of y in the figure for the sake of simplicity, since it is sufficient to bear in mind that the positive ordinates y extend from the axis of x *upwards*. Let OA be the original and OA' the new axis of x; then are OD, DP, and $OD', D'P$ the co-ordinates of P. Draw $D'H$ perpendicular and $D'K$ parallel to OA; and let the ordinates OD', $D'P$ which refer the point P to the new axes be $x'y'$. Then, assuming the given $\angle A'OA=\angle D'PK=\omega$, we shall have $OH=x'\cos\omega$ and $DH=KD'=y'\sin\omega$, the difference of which gives

$$OD=x=x'\cos\omega-y'\sin\omega\;\ldots\ldots(1);$$

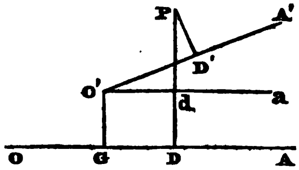
also $D'H=KD=x'\sin\omega$ and $PK=y'\cos\omega$, which added give

$$PD=y=x'\sin\omega+y'\cos\omega\;\ldots\ldots(2).$$

These values of OD and PD introduced instead of x and y will produce an expression involving $x'y'$ and the given angle ω.

It must be here observed that the new axis OA' of x is taken on that side of OA on which the ordinates y are positive; when taken on the contrary or under side of OA, the angle ω will have a negative value.

45. *An expression involving the co-ordinates of a point related to two rectangular axes being given to deduce the corresponding equivalent in terms of the ordinates of the same point referred to two other rectangular axes making a given angle with the former and having a different origin.*



Let $O'A'$ be the new axis of x; PD' perpendicular to it from the point P, and $O'a$ parallel to OA. Denote the position of the new origin O' by ab, viz: $OG = a$, $GO' = b$; let the position of P with respect to the new axes be $x'y'$, that is $O'D' = x'$, $D'P = y'$; and, as before, denote the given angle of inclination $A'O$a by ω.

Then, (44,) the co-ordinates of P referred to $O'a$ as an axis of x are

$$O'd = x' \cos\omega - y' \sin\omega;\ Pd = x' \sin\omega + y \cos\omega;$$

and hence, (43,) the values of the original ordinates are

$$OD = x = x' \cos\omega - y' \sin\omega + a \ldots (1),$$

$$PD = y = x' \sin\omega + y' \cos\omega + b \ldots . (2).$$

By substituting these instead of x and y in the proposed expression we shall get an expression, involving $x'y'$ with the new additional constants $a\,b$ and the given angle ω, which will be the one required.

SECTION V.

EQUATIONS OF THE SECOND DEGREE.

46. The general form for equations of the second degree, being those in which the ordinates $x\,y$ are involved to the second power, is

$$A\,x^2+B\,y^2+C\,x\,y+a\,x+b\,y+c=o$$

wherein each of the constants A, B, C, a, b, c, may be either *positive* or *negative*.

Let us in the first place transfer the equation to two other rectangular axes parallel to the original ones and having their origin at a point whose ordinates are $a\,b$; and, (43,) by substituting $x+x'$ and $y+y'$ for x and y, we shall find the corresponding equation to be

$$A\,(x^2+2\,x'\,x+x'^2)+B\,(y^2+2\,y'\,y+y'^2)$$
$$+C\,(x\,y+y'\,x+x'\,y+x'\,y')$$
$$+a\,(x+x')+b\,(y+y')+c=o;$$

which arranged for x and y becomes

$$Ax^2+By^2+Cxy$$
$$+(2Ax'+Cy'+a)\,x+(2By'+Cx'+b)\,y$$
$$+(Ax'^2+By'^2+Cx'\,y'+a\,x'+b\,y'+c)=o.$$

47. The first three co-efficients A, B, C stand unaffected with the new constants x', y', by which we observe that they are independent of the position of the origin; and hence the position of the origin of any equation of the second degree depends entirely on the values of the three last co-efficients $a\ b, c$.

48. We may now assume the values of the two ordinates $x'\,y'$ at pleasure since the position of the new origin is entirely arbitrary; and consequently, by the principles of algebra, we may fulfil any two possible conditions which involve them; let us therefore put the coefficients of x and y each equal to nothing, viz:

$$2Ax' + Cy' + a = o,$$
$$2By' + Cx' + b = o;$$

and thence

$$x' = \frac{Cb - 2Ba}{4AB - C^2}, y' = \frac{Ca - 2Ab}{4AB - C^2};$$

hence also, by substitution, the last term

$$Ax'^2 + By'^2 + Cx'y' + ax' + by' + c = \frac{Cab - Ab^2 - Ba^2}{4AB - C^2} + c;$$

or by assuming

$$Cab - Ab^2 - Ba^2 + c(4AB - C^2) = G,$$

it becomes =

$$\frac{G}{4AB - C^2}.$$

The equation is thus transformed into

$$Ax^2 + By^2 + Cxy + \frac{G}{4AB - C^2} = o \ldots (a),$$

in which the fourth and fifth terms are wanting.

49. Let us now transfer this equation to two other rectangular axes inclined at an angle ω with the former and retaining the same origin; and, (44,) substituting $x \cos \omega - y \sin \omega$ and $x \sin \omega + y \cos \omega$ for x and y, we get for the corresponding equation

$$\begin{aligned} &A(x^2 \cos^2 \omega + y^2 \sin^2 \omega - 2xy \cos \omega \sin \omega) \\ &+ B(x^2 \sin^2 \omega + y^2 \cos^2 \omega + 2xy \cos \omega \sin \omega) \\ &+ C\left\{x^2 \cos \omega \sin \omega - y^2 \cos \omega \sin \omega + xy(\cos^2 \omega - \sin^2 \omega)\right\} \\ &+ \frac{G}{4AB - C^2} = o, \end{aligned}$$

which arranged for x and y, observing that

$$\cos^2 \omega - \sin^2 \omega = \cos 2\omega \text{ and } 2 \cos \omega \sin \omega = \sin 2\omega,$$

becomes

$$\begin{aligned} &(A \cos^2 \omega + B \sin^2 \omega + C \cos \omega \sin \omega) x^2 + (A \sin^2 \omega + B \cos^2 \omega \\ &- C \cos \omega \sin \omega) y^2 \\ &+ \left\{C \cos 2\omega - (A - B) \sin 2\omega\right\} xy \\ &+ \frac{G}{4AB - C^2} = o, \end{aligned}$$

By taking the value of ω so as to exterminate xy,

$$C \cos 2\omega - (A - B) \sin 2\omega = o$$

$$\text{and } \tan 2\omega = \frac{C}{A - B},$$

which reduces the equation to

$$(A \cos^2 \omega + B \sin^2 \omega + C \cos \omega \sin \omega)\, x^2 + (A \sin^2 \omega + B \cos^2 \omega - C \cos \omega \sin \omega)\, y^2$$

$$+ \frac{G}{4AB - C^2} = o;$$

and it hence appears that every line of the second order may be referred to two determinate rectangular axes so that its equation shall be transformed into the above form. By assuming

$$A \cos^2 \omega + B \sin^2 \omega + C \cos \omega \sin \omega = A'',$$

$$A \sin^2 \omega + B \cos^2 \omega - C \cos \omega \sin \omega = B'',$$

it becomes

$$A''\, x^2 + B''\, y^2 + \frac{G}{4AB - C^2} = o \ldots (b).$$

50. Now if the principal semi-diameters of an ellipse and hyperbola be denoted by a', b', and the former be taken for the axis of x and the origin at the centre, their equations will be as follow:

For the ellipse

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \text{ or } b'^2x^2 + a'^2y^2 - a'^2b'^2 = o;$$

and for the hyperbola

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = \pm 1, \text{ or } b'^2x^2 - a'^2y^2 \mp a'^2b'^2 = o,$$

the under sign representing the conjugate hyperbola.

The signs may be all changed if necessary.

By means of these two equations and the foregoing transformed equation, (b,) we deduce the following particulars relative to the general equation.

51. 1st. When A'', B'' are both negative and G, $4AB - C^2$ have the same sign, the equation determines an *ellipse;* and

when A'', B'' are both of them positive and G and $4AB - C^2$ have different signs, the locus is also an *ellipse*.*

52. 2nd. When A'', B'' are of different signs and G not $= o$, the locus is *an hyperbola.*

53. 3rd. In each of these cases the squares of the principal semi-diameters are equal to

$$\frac{\pm G}{A''(4AB - C^2)}, \quad \frac{\pm G}{B''(4AB - C^2)},$$

the under sign being for the ellipse and either sign for the hyperbola.

54. 4th. The values of G, A'', B'' are determined from the equations

$$G = Cab - Ab^2 - Ba^2 + c(4AB - C^2) \ldots (1),$$

$$\tan 2\omega = \frac{C}{A - B} \ldots (2),$$

$$\left.\begin{array}{l} A'' = A\cos^2\omega + B\sin^2\omega + C\cos\omega\sin\omega, \\ B'' = A\sin^2\omega + B\cos^2\omega - C\cos\omega\sin\omega; \end{array}\right\} \ldots (3)$$

wherein ω is the angle included between the original axis of x and the principal diameter of the curve.

55. 5th. The position of the centre of the curve is determined by

$$x' = \frac{Cb - 2Ba}{4AB - C^2}, \quad y' = \frac{Ca - 2Ab}{4AB - C^2}.$$

56. 6th. When the equation is of the form

$$Ax^2 + By^2 + Cxy + c = o,$$

wherein the fourth and fifth terms of the general equation are wanting, we have $a = o$, $b = o$ and thence $x' = o$, $y' = o$ which therefore shews the origin to be at the centre of the curve. This agrees with equation (*b*,) article 49, where the origin is transferred to the centre.

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* For the immediate values of A'', B'' see article 73.

57. 7th. By adding the equations (3), article 54, we find

$$A'' + B'' = A + B.$$

Hence we see that, whatever be the position of the axes of co-ordinates, the sum of the co-efficients of x^2 and y^2 will be the same.

58. 8th. When $G = o$ and also A'' and B'' of different signs, the general equation defines a straight line.
For in this case the transformed equation (*b*), article 49, becomes

$$A'' x^2 + B'' y^2 = o,$$

which gives

$$\frac{y}{x} = \sqrt{-\frac{A''}{B''}};$$

and this value is real when A'', B'' have different signs.

59. 9th. In the two following cases it will be found that no real values of x and y can possibly fulfil the equation (*b*); and consequently that the equation can have no locus.
First. When G and $4AB - C^2$ are of the same sign and A'', B'' both of them positive.
Second. When G and $4AB - C^2$ are of different signs and A'', B'' are both negative.

60. 10th. When $G = o$ and A'', B'' have the same sign, no real values of x and y can satisfy the equation (*b*,) except the particular case of $x = o, y = o$. In this case therefore the locus is the single point corresponding with the new origin $x' y'$.

61. 11th. It appears that by changing the position of the origin to the centre $x' y'$
the equation

$$A x^2 + B y^2 + C x y + a x + b y + c = o$$

is transformed into the form

$$A x^2 + B y^2 + C x y + h = o,$$

wherein $h = \frac{G}{4 AB - C^2}$.

Also, that by taking two other axes of co-ordinates making an angle with these so that $\tan 2\omega = \frac{C}{A - B}$, the equation

$$A x^2 + B y^2 + C x y + h = o$$

becomes of the form

$$A'' x^2 + B'' y^2 + h = o$$

wherein $A'' + B'' = A + B$ and the constant h is unchanged.

62. 12th. Let x'', y'' be the two semi-diameters of the curve

$$A x^2 + B y^2 + C x y + h = o$$

which coincide with the axes of co-ordinates to which it is referred, and they will be determined by taking first $y = o$ and then $x = o$ in the equation, the results being

$$x''^2 = -\frac{h}{A},\ y''^2 = -\frac{h}{B}.$$

Let also a', b' be the principal semi-diameters which coincide with the axes to which the equation

$$A'' x^2 + B'' y^2 + h = o$$

appertains; and we similarly have

$$a'^2 = -\frac{h}{A''},\ b'^2 = -\frac{h}{B''}.$$

Hence as $A'' + B'' = A + B$, we have

$$\frac{1}{x''^2} + \frac{1}{y''^2} = \frac{1}{a'^2} + \frac{1}{b'^2};$$

That is the sum of the reciprocals of the squares of any two semi-diameters, of a curve of the second order, which are perpendicular to each other, is the same; and, in reference to the general equation, is =

$$-\frac{A+B}{h} = -\frac{A+B}{G}(4 A B - C^2).$$

63. When $4AB - C^2 = o$, we have, (55), $x'y'$ both of them *infinite* which shews the centre of the curve to be infinitely remote from the origin. It becomes hence necessary to consider this case separately.

Let

$$Ax^2 + By^2 + Cxy + ax + by + c = o$$

be the general equation in which $4AB - C^2 = o$.

Then, transferring the origin to a point $x'y'$, the corresponding equation, (46,) is

$$\begin{gathered} Ax^2 + By^2 + Cxy \\ + (2Ax' + Cy' + a)x + (2By' + Cx' + b)y \\ + (Ax'^2 + By'^2 + Cx'y' + ax' + by' + c) = o. \end{gathered}$$

Let $x'y'$ determine some point in the curve, so that

$$Ax'^2 + By'^2 + Cx'y' + ax' + by' + c = o,$$

and the equation becomes

$$\begin{gathered} Ax^2 + By^2 + Cxy \\ + (2Ax' + Cy' + a)x + (2By' + Cx' + b)y = o. \end{gathered}$$

But, since $4AB - C^2 = o$ and $\because C = 2\sqrt{AB}$, we have

$$Ax^2 + By^2 + Cxy = (x\sqrt{A} + y\sqrt{B})^2.$$

Hence the reduced equation is equivalent to

$$\begin{gathered} (x\sqrt{A} + y\sqrt{B})^2 \\ + (2Ax' + Cy' + a)x + (2By' + Cx' + b)y = o. \end{gathered}$$

64. We shall now, as in article 49, transfer this equation to two other rectangular axes proceeding from the same origin and making an angle ω with the former; and, (44,) putting $x \cos\omega - y \sin\omega$ and $x \sin\omega + y \cos\omega$ for x and y, the resulting equation is

$$\begin{aligned} &\{(\cos\omega\sqrt{A} + \sin\omega\sqrt{B})x - (\sin\omega\sqrt{A} - \cos\omega\sqrt{B})y\}^2 \\ &+ \{(2Ax' + Cy' + a)\cos\omega + (2By' + Cx' + b)\sin\omega\}x \\ &- \{(2Ax' + Cy' + a)\sin\omega - (2By' + Cx' + b)\cos\omega\}y = o. \end{aligned}$$

Let ω satisfy the condition

$$\cos\omega\sqrt{A} + \sin\omega\sqrt{B} = o,$$

which will give

$$\tan\omega = -\sqrt{\frac{A}{B}},\ \cos\omega = -\sqrt{\frac{B}{A+B}},\ \sin\omega = \sqrt{\frac{A}{A+B}};$$

and thence

$$\sin\omega\sqrt{A}-\cos\omega\sqrt{B}=\sqrt{(A+B)};$$

$$(2Ax'+Cy'+a)\cos\omega+(2By'+Cx'+b)\sin\omega=-\frac{a\sqrt{B}-b\sqrt{A}}{\sqrt{(A+B)}}$$

and $(2Ax'+Cy'+a)\sin\omega-(2By'+Cx'+b)\cos\omega=$

$$2x'\sqrt{A(A+B)}+2y'\sqrt{B(A+B)}+\frac{a\sqrt{A}+b\sqrt{B}}{\sqrt{(A+B)}}$$

$$=2\sqrt{(A+B)}\left\{x'\sqrt{A}+y'\sqrt{B}+\frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}\right\}.$$

The equation thus becomes

$$(A+B)y^2-\frac{a\sqrt{B}-b\sqrt{A}}{\sqrt{(A+B)}}x$$

$$-2\sqrt{(A+B)}\left\{x'\sqrt{A}+y'\sqrt{B}+\frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}\right\}y=o.$$

65. We have, (63,) assumed $x'y'$ to determine a point in the curve, but not restricted ourselves to any particular point; we may therefore take this point where the curve is intersected by a straight line whose equation is

$$x\sqrt{A}+y\sqrt{B}+\frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}=o,$$

by means of which we shall have

$$x'\sqrt{A}+y'\sqrt{B}+\frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}=o,$$

which reduces the equation to

$$(A+B)y^2-\frac{a\sqrt{B}-b\sqrt{A}}{\sqrt{(A+B)}}.x=o,$$

$$\text{or } y^2-\frac{a\sqrt{B}-b\sqrt{A}}{(A+B)^{\frac{3}{2}}}.x=o\ldots.(c).$$

But the equation of a parabola, whose parameter is p, taking the origin at the vertex and the principal axis for the axis of x, is

$$y^2=px \text{ or } y^2-px=o.$$

Hence the following particulars:—

66. 1st. When $a\sqrt{B}-b\sqrt{A}$ *not* $=o$, the locus is a *Parabola* whose parameter is equal to

$$\frac{a\sqrt{B}-b\sqrt{A}}{(A+B)^{\frac{3}{2}}}.$$

67. 2nd. According to article 12, the equation

$$x\sqrt{A}+y\sqrt{B}+\frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}=o$$

defines a straight line inclined to the original axis of x at an angle whose tangent $=-\sqrt{\frac{A}{B}}$ and which is therefore equal to ω, the inclination of the axis of the curve, with the axis of x; this line, (65,) also passing through the vertex $x'y'$, it must coincide with the axis of the curve. Therefore the above equation properly represents the *principal diameter* of the curve; by uniting it with the original equation we may hence find the co-ordinates $x'y'$ of its intersection with the curve, or the vertex.

68. 3rd. If $a\sqrt{B}-b\sqrt{A}=o$, or $a\sqrt{B}=b\sqrt{A}$, the equation (c) gives simply

$$y=o,$$

which shews the locus in this case to be a straight line corresponding with the new axis of x, the equation of which is given, (67).

69. 4th. The equation $4AB-C^2=o$ giving $C^2=\pm 2\sqrt{AB}$, the values of the constants A, B must have the same sign to make C real, that is, they must be either both of them positive or both negative; and hence we may consider them both positive for, when negative, they can be made so by preliminarly changing all the signs of the original equation. If, under this consideration, C be negative we shall have $C=-2\sqrt{AB}$ instead of $+2\sqrt{AB}$; in this case, the foregoing operations hold good by either substituting $-\sqrt{A}$ instead of $\sqrt{A}$ or $-\sqrt{B}$ for $\sqrt{B}$, or by considering either $\sqrt{A}$ or $\sqrt{B}$ to have a negative value; and $\tan\omega$ will become hence $=+\frac{A}{B}$ instead of $-\sqrt{\frac{A}{B}}$.

Thus we see that, when C is *negative*, *tan* ω is *positive* and $\omega < \frac{\pi}{2}$; and that, when C is *positive*, *tan* ω is *negative* and $\therefore \omega > \frac{\pi}{2}$.

The foregoing investigations lead immediately to the solutions of the three following propositions:

70. *To express the equations of the principal diameters of a curve of the second order which is determined by the general equation.*

The co-ordinates of the centre, (55,) are

$$x' = \frac{Cb - 2Ba}{4AB - C^2},\ y' = \frac{Ca - 2Ab}{4AB - C^2}.$$

Let ω denote the inclination of one of the principal diameters of the curve with the co-ordinate axis of x; and, (54,)

$$\tan 2\omega = \frac{C}{A - B},$$

from which

$$\tan \omega = \frac{\sec 2\omega - 1}{\tan 2\omega} = \frac{\sqrt{\{(A-B)^2 + C^2\}} - (A - B)}{C}.$$

Now the diameter being inclined to the co-ordinate axis of x at the angle ω and also passing through the centre $x'y'$ of the curve, its equation, (22,) is

$$y - y' = (x - x') \tan \omega.$$

Hence by substitution we have

$$y - \frac{Ca - 2Ab}{4AB - C^2} =$$

$$\left(x - \frac{Cb - 2Ba}{4AB - C^2}\right) \cdot \frac{\sqrt{\{(A-B)^2 + C^2\}} - (A - B)}{C} \ldots (x),$$

for the equation of one of the principal diameters.

The other diameter passing through the centre $x'y'$ perpendicular to this, its equation, (36,) is

$$y - \frac{Ca - 2Ab}{4AB - C^2} =$$

$$-\left(x-\frac{Cb-2Ba}{4AB-C^2}\right).\frac{C}{\sqrt{\{(A-B)^2+C^2\}}-(A-B)}$$

or, which is the same,

$$y-\frac{Ca-2Ab}{4AB-C^2}=$$

$$-\left(x-\frac{Cb-2Ba}{4AB-C^2}\right).\frac{\sqrt{\{(A-B)^2+C^2\}}+(A-B)}{C}\ldots(y).^*$$

71. *Cor.* 1. If the origin of the ordinates be the centre of the curve its equation, (56,) will be of the form

$$Ax^2+By^2+Cxy+c=o;$$

and we shall have $a=o$, $b=o$. In this case therefore the equations of the principal diameters are

$$y=\frac{\sqrt{\{(A-B)^2+C^2\}}-(A-B)}{C}.x$$

$$\text{and } y=-\frac{\sqrt{\{(A-B)^2+C^2\}}+(A-B)}{C}.x$$

72. *Note.* The equation

$$tan\,2\omega=\frac{C}{A-B}$$

applies equally to both diameters. For, if 2ω fulfil this equation, it will also hold good when $2\omega\pm\pi$ is substituted; and, ω denoting the inclination of one of the diameters, $\omega\pm\frac{\pi}{2}$ will evidently be that of the other.

From this equation we derive generally

$$tan\,\omega=\frac{sec\,2\omega-1}{tan\,2\omega}=\pm\frac{\sqrt{\{(A-B)^2+C^2\}}-(A-B)}{C},$$

the upper sign appertaining to one of the axes and the under sign to the other.

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[*] By uniting these equations of the principal diameters with the given equation of the curve we may thence find the positions of the vertices.

Thus, by making use of the under sign, the equation (x) will become the same as the equation (y), and vicè versâ ——, because

$$-\frac{\sqrt{\{(A-B)^2+C^2\}}-(A-B)}{C}\cdot\frac{\sqrt{\{(A-B)^2+C^2\}}-(A-B)}{C}$$
$$=-1.$$

When $4AB-C^2=o$, see article 67.

73. *The equation of a curve of the second order being given to find the values of its principal semi-diameters.*

The squares of the semi-diameters are, (53,) equal to

$$\frac{+G}{A''(4AB-C^2)},\quad\frac{+G}{B''(4AB-C^2)},$$

wherein, (54,)

$$G=Cab-Ab^2-Ba^2+c(4AB-C^2);$$
$$A''=A\cos^2\omega+B\sin^2\omega+C\cos\omega\sin\omega,$$
$$B''=A\sin^2\omega+B\cos^2\omega-C\cos\omega\sin\omega,$$
$$\text{and } \tan 2\omega=\frac{C}{A-B}.$$

From the last we deduce

$$\cos^2\omega=\frac{1}{2}\left(1+\frac{1}{\sec 2\omega}\right)=\frac{1}{2}\left(1+\frac{A-B}{\sqrt{\{(A-B^2)+C^2\}}}\right),$$

$$\sin^2\omega=\frac{1}{2}\left(1-\frac{1}{\sec 2\omega}\right)=\frac{1}{2}\left(1-\frac{A-B}{\sqrt{\{(A-B)^2+C^2\}}}\right),$$

$$\cos\omega\sin\omega=\frac{C}{2\sqrt{\{(A-B)^2+C^2\}}};$$

and hence we get

$$A''=\frac{A+B+\sqrt{\{(A-B)^2+C^2\}}}{2},$$

$$B''=\frac{A+B-\sqrt{\{(A-B)^2+C^2\}}}{2}.$$

These and the foregoing value of G substituted, the squares of the principal semi-diameters of the curve are found equal to

$$\pm \frac{2\{Cab - Ab^2 - Ba^2 + c(4AB - C^2)}{(4AB - C^2)\left[A + B + \sqrt{\{(A-B)^2 + C^2\}}\right]},$$

$$\pm \frac{2\{Cab^2 - Ab^2 - Ba^2 + c(4AB - C^2)\}}{(4AB - C^2)\left[A + B - \sqrt{\{(A-B)^2 + C^2\}}\right]},$$

the under sign being for the ellipse and either sign for the hyperbola, (53.)

74. When the origin is at the centre of the curve, (61,) $a = o, b = o$; and therefore in this case the squares of the principal semi-diameters are equal to

$$\frac{\pm 2c}{A+B+\sqrt{\{(A-B)^2+C^2\}}}, \quad \frac{\pm 2c}{A+B-\sqrt{\{(A-B)^2+C^2\}}}.$$

75. *To determine the particular description of a curve of the second order from the immediate relative values of the constants which belong to its equation.*

In (51), (52) and the subsequent articles, the different cases are severally stated, throughout the various relations of A'', B'', G, $4AB - C^2$, &c., where A'', B'' are, (54,) expressed in terms of the coefficients A, B, C, by means of the arc ω as a subsidiary. It is hence only necessary to transfer the relations of A'', B'' to those of the immediate coefficients A, B, C, which may be easily effected from their values which have already been found, (73,) viz:

$$A'' = \frac{A+B+\sqrt{\{(A-B)^2+C^2\}}}{2},$$

$$B'' = \frac{A+B-\sqrt{\{(A-B)^2+C^2\}}}{2}.$$

Thus it is evident that, when $(A+B)^2$ is greater than $(A-B)^2 + C^2$ the sign of $A+B$ cannot be affected with either the addition or subtraction of $\sqrt{\{(A-B)^2+C^2\}}$, and consequently that the values of A'', B'' will both have the same sign with $A+B$. But, when $(A+B)^2$ is greater than $(A-B)^2+C^2$, we shall have $(A+B)^2-\{(A-B)^2+C^2\} = 4AB-C^2$ *positive*. Hence, when $4AB-C^2$ is *positive* A'' and B'' will both of them have the same sign with $A+B$, that is, they will both be positive when $A+B$ is positive and both negative when $A+B$ is so.

It is also pretty obvious that, when $(A+B)^2$ is less than $(A-B)^2+C^2$, the values of A'' B'' will have different signs, that is, the one will be positive and the other negative.

In this case we shall have $(A+B)^2-\{(A-B)^2+C^2\} = 4AB-C^2$ *negative*. Thus we see, when $4AB-C^2$ is *negative*, that A'', B'' are of different signs.*

Again, under the class $4AB-C^2=o$, when the value of $a\sqrt{B}-b\sqrt{A}=o$, we shall have $2\sqrt{A}(a\sqrt{B}-b\sqrt{A}) = 2a\sqrt{AB}-2Ab=o$

$$\text{or } Ca-2Ab=o.$$

Hence also, when $a\sqrt{B}-b\sqrt{A}$ not $=o$, we shall have $Ca-2Ab$ not $=o$.

By carefully comparing these relations with the articles (51), (52), (58), (59), (60), (66), and (68), we find the different descriptions of the curve to be as in the following arrangement, wherein

$$G=Cab-Ab^2-Ba^2+c(4AB-C^2).$$

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* These relatious are also pretty evident from the equations

$$A''+B''=A+B,$$
$$4A''B''=4AB-C^2.$$

When $4AB - C^2$ is *positive*; and $A + B$ *and* G are of *different signs*, the locus is *an* ELLIPSE.

........................; .... are of *the same signs*, ... IMPOSSIBLE.

........................; ——————— *and* $G = o$, ... *a* POINT.

.... is *negative*; .... G *not* $= o$, ... *an* HYPERBOLA.

........................; ....$G = o$, ... *a* STRAIGHT LINE.

......$4AB - C^2 = o$ .....; and $Ca - 2Ab$ *not* $= o$, ... *a* PARABOLA.

........................; .... $Ca - 2Ab = o$, ... *a* STRAIGHT LINE.

Note.——When the origin of the axes is at the centre of the curve and consequently the equation is $Ax^2 + By^2 + Cxy + c = o$, the value of G is simply $c\,(4AB - C^2)$.

SECTION VI.

FORMULÆ FOR CURVES, &c. INVOLVING THE DIFFERENTIAL AND INTEGRAL CALCULUS.

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76. *The equation of a variable straight line being given to find the point of intersection of two of its positions which are indifinitely near to each other.*

Let $y = mx + h$ be the equation of the line.
It is plain that when m, h are given constants the straight line is given and fixed; thus, in order that the line may assume another position it is necessary that one or both of the characters m, h shall become of different values. Hence, under the circumstances of the proposition one or both of the values m, h must be subject to variation,
Let $x'y'$ be the co-ordinates of the required point which will, of course, fulfil the equation

$$y' = mx' + h.$$

If we now suppose the line to vary to an indefinitely near consecutive position, the point of intersection $x'y'$ being fixed during the change its ordinates x' and y' will hence remain invariable. Hence differentiating, and considering $x'y'$ as constant, we get

$$o = x'\,dm + dh,$$

from which and the above equation we find the required point to be

$$x' = -\frac{dh}{dm},\ y' = h - m\frac{dh}{dm}.$$

To compute these values it is necessary for h to be a function of m; this is, in fact, necessary to impose a law on the variation of the line, the position of which would otherwise be absolutely arbitrary,

77. *Cor.* 1. If the line be supposed in motion this point will obviously be the centre of instantaneous rotation; and its locus will evidently be that curve to which the line is

always a tangent* (see 81)——hence the nature of this curve may be found by eliminating the introduced variable from the values of x' and y'.

78. *Cor.* 2. Similarly to the foregoing may we find the point of intersection of two indefinitely near positions of a variable curve by differentiating its equation and considering x', y' as constant; the values of x', y' being determined from the given and the resulting equation.

79. *To find the area of a curve comprehended between two given values of y and the axis of x.*
By taking two values of y indefinitely near to each other, the space included between them may obviously be considered as rectangular and consequently as having a value $= ydx$.
Thus we have

$$\text{The area} = \int ydx.$$

This integral, between the limits $x'y'$, xy will give the area contained between the ordinates y' and y.

80. *To find the length of any portion of a curve from the equation between its rectangular co-ordinates.*
Let s denote the length of the curve corresponding with the ordinates $x\,y$ and reckoned from any given point, and we shall evidently have

$$ds^2 = dx^2 + dy^2;$$
$$\therefore ds = \sqrt{(dx^2 + dy^2)}$$
$$\text{and } s = \int \sqrt{(dx^2 + dy^2)}.$$

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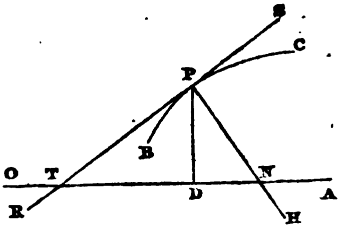
* This is rendered evident by considering it inversly; thus, by supposing a tangent to move over a curve line its successive indefinite intersections will obviously coincide with the points of contact and therefore trace out the same curve.

By taking this integral between the limits $x'y'$ and xy we find the length of the portion of the curve intercepted by those points.

81. *Def.* Let *BPC* be any curve.
Then a straight line *RS*, drawn so as to touch it at any point *P* is called a *tangent;* and the point *P* is called the point of contact.



Another straight line *PH*, drawn perpendicular to the curve at the point *P* and consequently also perpendicular to the tangent *RS*, is called a *normal* at that point.
By the length of a tangent we generally understand that portion which is limited by its intersection with the axis of x and the point of contact; the value of the normal is similarly understood to be that portion of it which is intercepted by the axis of x and the point *P*. Thus, *PT* is the *length* of the tangent and *PN* that of the normal.
OD and *DP* being the ordinates of *P*, *TD* is called the *subtangent* and *DN* the *subnormal.*

82. *The equation of a curve being given to find the equation of the tangent at any point.*

Let $x'y'$ be the co-ordinates of the point of contact and ω the inclination of the tangent with the axis of x; then, (22,) the equation is

$$y - y' = (x - x') \tan \omega;$$

but we obviously have

$$\tan \omega = \frac{dy'}{dx'}$$

$\therefore$ the required equation is

$$y - y' = \frac{dy'}{dx'} (x - x')$$

$$\text{or } (x - x') - \frac{dx'}{dy'} (y - y') = 0,$$

wherein $x'y'$ is the point of contact and xy any point whatever in the tangent.

Or it may be expressed differentially

$$dy' (x - x') - dx' (y - y') = o.$$

83. *Cor.* 1. Hence the equation to the *normal* through the same point, (36,) is

$$y - y' = -\frac{dx'}{dy'} (x - x')$$

$$\text{or } (x - x') + \frac{dy'}{dx'} (y - y') = o.$$

This may also be expressed differentially

$$dx' (x - x') + dy' (y - y') = o.$$

84. *Cor.* 2. The equation of the tangent at the point $x'y'$ being

$$dy' (x - x') - dx' (y - y') = o$$

$$\text{or } dy' . x - dx' . y + (y' dx' - x' dy') = o,$$

the perpendicular drawn to it from the origin, (38,) is equal to

$$\frac{x' dy' - y' dx'}{\sqrt{(dx'^2 + dy'^2)}} = \pm \frac{x' dy' - y' dx'}{ds'}.$$

85. *Cor.* 3. Similarly, the perpendicular from the origin upon the normal is equal to

$$\frac{x' dx' + y' dy'}{\sqrt{(dx'^2 + dy'^2)}} = \pm \frac{x' dx' + y' dy'}{ds'}.$$

86. *Cor.* 4. By taking $y = o$ in the equation of the tangent, x will determine its intersection with the axis of x and hence in this case $x' - x$ is the subtangent at the point $x'y'$.

$$\because \textit{Subtangent} = x' - x = \frac{y' dx'}{dy'}.$$

By similarly taking $y = o$, in the equation of the normal, we find

$$\textit{The Subnormal} = x' - x = \frac{y' dy'}{dx'}.$$

Hence also

$$\textit{the Tangent} = \sqrt{\left(y'^2 + \frac{y'^2\,dx'^2}{dy'^2}\right)} = \frac{y'\,ds'}{dx'},$$

$$\text{and } \textit{the Normal} = \sqrt{\left(y'^2 + \frac{y'^2\,dy'^2}{dx'^2}\right)} = \frac{y'\,ds'}{dx'},$$

s' being the arc of the curve corresponding with the point $x'y'$.

ASYMPTOTES.

87. Two curves or a curve and straight line are said to be asymptotic when they continually approach nearer and nearer to each other but do not meet at any finite distance.

By an asymptote to a curve we generally understand a straight line such that if it and the curve be indefinitely continued they will continually approach each other but never meet; or it may be considered as a tangent to the curve when the point of contact is at an infinite distance.

In the equation of the tangent let $y = o$ and we shall find the intercept of the axis of x, between the origin and the tangent at the point $x'y'$, to be

$$x = x' - \frac{y'\,dx'}{dy'} = \frac{x'\,dy' - y'\,dx'}{dy'}.$$

Again by taking $x = o$ we similarly find the intercept of the axis of y, between the origin and the tangent, to be

$$y = y' - \frac{x'\,dy'}{dx'} = \frac{y'\,dx' - x'\,dy'}{dx'},$$

If, when $x' = \infty$ or $y' = \infty$, either of these values of x and y are finite, the curve has asymptotes which will thence be determined.

When x is *finite*, but y *infinite*, the asymptote is parallel to the axis of y,

When y is *finite*, but x *infinite*, the asymptote is parallel to the axis of x.

But when the values of x and y are *both of them infinite*, the asymtote is at an infinite distance from the origin. In this case the curve is said to have no asymptote.

When the values of x and y are *both* $= o$ the asymptote passes through the origin and its position must be determined from the value of $\frac{dy'}{dx'}$, when $x' = \infty$ or $y' = \infty$.

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88. *The equation of a curve being given to find whether, at a particular point, it is convex or concave to the axis of* x. As in article 82, let $x'y'$ denote the ordinates of the proposed point in the curve and ω the inclination of the tangent at that point with the axis of x, and

$$\frac{dy'}{dx'} = \tan \omega,$$

Now when the curve at the point $x'y'$ is *concave* to the axis of x the angle ω will evidently, if x increase, *decrease* when y is *positive* and *increase* when y is *negative*; and therefore $d.\frac{dy'}{dx'}$ will have a sign *contrary* to that of y. But when the curve is *convex* to the axis of x the inclination of the tangent will obviously, when x increases, *increase* or *decrease* accordingly as y is *positive* or *negative* and consequently $d.\frac{dy'}{dx'}$ will have the *same* sign with y'.

Hence, taking dx' *constant*, the curve at the point $x'y'$ will be *concave* towards the axis of x when $d^2 y'$ has a *contrary sign with* y'; and it will present a *convex* side towards the axis of x when $d^2 y'$ has *the same sign* with y'. Or, which amounts to the same, the curve is *convex* or *concave* to the axis of x accordingly as $y' d^2 y'$ is *positive* or *negative*.

89. *Cor.* 1. Hence also, by supposing dy' *constant*, the curve at the point $x'y'$ will be *convex* or *concave* to the axis of y accordingly as $d^2 x'$ has *the same* or *a different sign* with x'; or it will be *convex* or *concave* towards the axis of y accordingly as $x' d^2 \varphi'$ is *positive* or *negative*.

90. *Cor.* 2. When a curve is first convex and becomes afterwards concave to the axis of x it must have passed a

point of contrary flexure ———— in this case, supposing dx' constant, d^2y' will hence experience a change of sign; and the *point of contrary flexure* will evidently be where $d^2y' = o$.

91. *The equation of a curve being given to find the radius of curvature, or the radius of that circle which touches it most intimately at any given point.*

A tangent to any curve may be conceived to be a straight line drawn through two of its points which are indefinitely near to each other; and hence the first differentials of the ordinates which appertain to the tangent must correspond with those of the curve at the point of contact.
Similarly may we conceive the *osculating circle* or the *circle of curvature* to be that circle which passes through three successive points of the curve which are indefinitely near to each other; in this case, therefore, both the first and second differentials of the ordinates which belong to the circle and curve must correspond at the point of contact.
Let $x''y''$ be the co-ordinates of the centre of the circle, and we shall have $x - x''$, $y - y''$ for the two lines drawn from it respectively parallel to x and y and terminating in the circumference at the point of contact; hence, denoting its radius by ϱ, its equation, (25,) is

$$(x - x'')^2 + (y - y'')^2 = \varrho^2.$$

Now since, as has been observed, this circle corresponds with the curve at two other points contiguous to the point of contact, we may differentiate twice and consider the first and second differentials of the ordinates xy as agreeing with those of the curve. Hence differentiating, observing that $x''y''$ are invariable, we get

$$dx\,(x - x'') + dy\,(y - y'') = o,$$
$$d^2x\,(x - x'') + d^2y\,(y - y'') + ds^2 = o;$$

where

$$ds^2 = dx^2 + dy^2,$$

s being the length of the curve.

From these two equations we deduce

$$x - x'' = \frac{-dy\,ds^2}{dy\,d^2x - dx\,d^2y},\quad y - y'' = \frac{dx\,ds^2}{dy\,d^2x - dx\,d^2y}.$$

Hence we find

$$\varrho^2 = (x - x'')^2 + (y - y'')^2 = \frac{ds^6}{(dy\,d^2x - dx\,d^2y)^2},$$

$$\therefore \varrho = \pm \frac{ds^3}{dy\,d^2x - dx\,d^2y}.$$

In this expression for the radius of curvature we may assume an independent variable at pleasure.

92. It may be otherwise very clearly determined by conceiving the centre of the circle of curvature to be the intersection of two normals drawn from two points of the curve which are indefinitely near to each other. Let a normal be drawn from the point xy and also another normal from a contiguous point, the intercept of the curve between these points being ds. Let also two tangents be drawn at these points, the former making an angle ω with the axis of x. Then the angle $\mp d\omega$ included by the tangents will evidently be the same as that included by the normals; and as the normals, which are radii of the osculating circle, subtend the arc ds of the curve, we obviously have

$$\mp \varrho\, d\omega = ds$$

$$\text{and } \therefore \varrho = \mp \frac{ds}{d\omega}.$$

But

$$\tan \omega = \frac{dy}{dx};$$

$$\therefore d\omega = \frac{d\tan\omega}{1 + \tan^2\omega} = \frac{d\dfrac{dy}{dx}}{1 + \dfrac{dy^2}{dx^2}} = \frac{dx^2 .\, d\dfrac{dy}{dx}}{ds^2}.$$

Therefore by substitution

$$\varrho = \mp \frac{ds^3}{dx^2 .\, d\dfrac{dy}{dx}} = \pm \frac{ds^3}{dy\,d^2x - dx\,d^2y}.$$

93. Since

$$dx^2+dy^2=ds^2,$$

we have, differentiating

$$dx\,d^2x+dy\,d^2y=ds\,d^2s$$

$$\therefore\ o=(dx\,d^2x+dy\,d^2y)^2-(ds\,d^2s)^2.$$

Adding the square of $dy\,d^2x-dx\,d^2y$ to this, the result is

$$(dy\,d^2x-dx\,d^2y)^2=ds^2\left\{(d^2x)^2+(d^2y)^2-(d^2s)^2\right\},$$

and hence

$$\varrho=\frac{ds^2}{\sqrt{\left\{(d^2x)^2+(d^2y)^2-(d^2s)^2\right\}}}.$$

By taking s for the independent variable, this gives

$$\varrho=\frac{ds^2}{\sqrt{\left\{(d^2x)^2+(d^2y)^2\right\}}}.$$

94. By taking x for the independent variable, or supposing dx to be constant,

$$\varrho=\frac{ds^3}{dy\,d^2x-dx\,d^2y}$$

becomes simply

$$\varrho=\frac{-ds^3}{dx\,d^2y}.$$

95. From the equations

$$x-x''=\frac{-dy\,ds^2}{dy\,d^2x-dx\,d^2y},\quad y-y''=\frac{dx\,ds^2}{dy\,d^2x-dx\,d^2y}$$

we find the position of the centre of the circle of curvature to be

$$x''=x+\frac{dy\,ds^2}{dy\,d^2x-dx\,d^2y},$$

$$y''=y-\frac{dx\,ds^2}{dy\,d^2x-dx\,d^2y};$$

or, supposing dx constant,

$$x''=x-\frac{dy\,ds^2}{dx\,d^2y},$$

$$y''=y+\frac{dx\,ds^2}{dx\,d^2y}.$$

By means of these and the equation of the curve, the ordinates xy and their differentials may be eliminated; and an equation will thence be found expressing the relation between x'' and y'', which will hence define the locus of the centre of the osculating circle for all points in the curve. This locus is denominated the *evolute* of the curve; and, on the contrary, the curve is called its *involute*.

96. As the centre of the osculating circle may be conceived to be the point of intersection of two normals which are indefinitely near to each other, it is obvious that the normal at any point of the curve must be a tangent to the evolute. See article 77.

Thus we see that a tangent drawn to the evolute at any point coincides with the radius of the osculating circle which is drawn to the point of contact.

97. The equation of this tangent, (62,) gives

$$dy''(x-x'')-dx''(y-y'')=o.$$

Let us now differentiate the equation

$$(x-x'')^2+(y-y'')^2=\varrho^2,$$

supposing $x''y''$ to vary, and we have

$$(dx-dx'')(x-x'')+(dy-dy'')(y-y'')=\varrho d\varrho;$$

but, $x''y''$ appertaining to the normal of the curve at the point xy, we have by its equation

$$dx(x-x'')+dy(y-y'')=o,$$

which rejected and the signs changed, we get

$$dx''(x-x'')+dy''(y-y'')=-\varrho d\varrho.$$

From this and the preceding equation of the tangent to the evolute we find

$$x-x''=-\varrho d\varrho\,.\,\frac{dx''}{ds''^2},\quad y-y''=-\varrho d\varrho\,.\,\frac{dy''}{ds''^2},$$

wherein

$$ds''^2=dx''^2+dy''^2,$$

s'' being the arc of the evolute from any given point.

These values of $x-x''$ and $y-y''$ substituted in the equation

$$(x - x'')^2 + (y - y'')^2 = \rho^2$$

give

$$\rho^2 \cdot \frac{d\rho^2}{ds''^2} = \rho^2$$

$$\therefore ds''^2 = d\rho^2$$

$$\because ds'' = \pm d\rho.$$

The integral of this gives

$$s'' = \rho \backsim \rho',$$

where ρ' is the radius of curvature corresponding with the given point from which s'' is estimated.

Hence we find the length of the arc of the evolute between any two points to be equal to the difference between the radii of the corresponding osculating circles.

98. By means of this principle we discover that the curve may be described by the unwinding of an inextensible thread from off the evolute. Thus let p be any point in the curve and pp'' the normal or radius of curvature touching the evolute at the point p''; then, this line pp'' being conceived to be a thread extending round the evolute, it is obvious, from the above property, that by unwinding this thread, keeping pp'' always stretched, the point p will trace out the curve.

Considering the evolute as a *curve*, its *involute* is thus described.

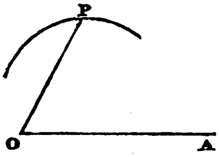
99. It appears, from the foregoing, that any given curve can have but one evolute, but may have an indefinite number of involutes as the value of pp'' at any point p'' is indeterminate. Hence, for any particular involute, the value of pp'' must be known at a given point p''.

100. The evolute being given, the equation of its involutes may be found by means of the values of x'', y'', (95,) in terms of xy and their differentials.

SECTION VII.

FORMULÆ APPERTAINING TO POLAR EQUATIONS.

101. Besides the application of co-ordinate axes there is another method of rendering the relative positions of points and the properties, &c. of curve lines, in the same plane, subject to the power of the algebraic analysis, by means of what is usually called a *polar equation.*



Thus a given indefinite right line OA, originating at O is denominated the *axis*; the fixed point O is denominated the *pole* or *origin*; any variable right line OP drawn to a point P is called a *radius vector*, to that point, and its angle of inclination POA, with the axis, the *polar angle.*

The radius vector OP we shall denote by r, and the polar angle POA by φ.

102. The polar equation of a curve is thus expressed

$$F(r, \varphi) = o;$$

and, in most cases, r may be separated so as to give

$$r = f(\varphi),$$

F and f denoting given trigonometrical functions.

The characters r, φ thus rendered subject to an equation, we shall evidently have particular values of r for each successive value of φ; and hence the point P will be limited to a particular curve, determined by the nature of the equation.

103. Particular values of r, φ, or such as belong to given, and, of course, invariable points are thus distinguished, $r'\varphi'$, $r''\varphi''$, &c. and the points to which they appertain are usually called the points $r'\varphi'$, $r''\varphi''$, &c.

104. It is often useful to transform expressions, involving rectangular co-ordinates, into their equivalents in terms of the

radius vector and the polar angle and conversely. This may be effected by means of the two following propositions.

105. *To reduce any expression or formula, involving the rectangular co-ordinates xy of any point and their differentials, into one involving the radius vector and the polar angle.*

By taking the axis of x for the polar axis, and the origin for the pole, we shall obviously have

$$x = r \cos \varphi,\ y = r \sin \varphi;$$

and hence also, by differentiation,

$$dx = dr \cos \varphi - rd\varphi \sin \varphi,$$
$$dy = dr \sin \varphi + rd\varphi \cos \varphi;$$
$$d^2x = d^2r \cos \varphi - 2\, dr\, d\varphi \sin \varphi - rd\varphi^2 \cos \varphi - rd^2\varphi \sin \varphi,$$
$$d^2y = d^2r \sin \varphi + 2\, dr\, d\varphi \cos \varphi - rd\varphi^2 \sin \varphi + rd^2\varphi \cos \varphi;$$

&c. &c. &c.

These values substituted in the given expression, will produce its equivalent in terms of r, φ and their differentials.

By supposing $d\varphi$ to be constant, the terms wherein $d^2\varphi$ occur will disappear.

106. If it be required to have the pole at a given point $x'y'$, we may previously transfer the origin of the rectangular axes to that point, by article 43. Or we may substitute

$$x = r \cos \varphi + x',\ y = r \sin \varphi + y',$$

and $x'y'$ being constant, the values of dx, dy, d^2x, d^2y, &c. as above.

And, if the polar axis be required to make an angle ω with the axis of x, we must obviously substitute $\varphi + \omega$ instead of φ.

107. *To reduce any expression, involving the radius vector and the polar angle, into one involving rectangular co-ordinates.*

By taking the polar axis for the axis of x, and the pole for the origin of xy, we shall have, from the foregoing equations,

$$r = \sqrt{(x^2 + y^2)};$$

also

$$\cos\varphi = \frac{x}{r} = \frac{x}{\sqrt{(x^2+y^2)}},\ \sin\varphi = \frac{y}{r} = \frac{y}{\sqrt{(x^2+y^2)}},$$

$$\tan\varphi = \frac{y}{x}\ \&c.,$$

or

$$\varphi = \cos^{-1}\frac{x}{\sqrt{(x^2+y^2)}} = \sin^{-1}\frac{y}{\sqrt{(x^2+y^2)}} = \tan^{-1}\frac{y}{x}\ \&c.,$$

wherein $\cos^{-1}\frac{x}{\sqrt{(x^2+y^2)}}$ signifies the arc whose *cosine* is $\frac{x}{\sqrt{(x^2+y^2)}}$, &c,

The required transformation may be accomplished by the substitution of these values; and the origin may afterwards be transferred to any given point, (43).

108. *The polar equation of a curve being given to find the length of any arc of it.*

By referring the points of the curve to rectangular co-ordinate axes, we have, (80,)

$$ds^2 = dx^2 + dy^2.$$

Hence, substituting the values of dx, dy, (105,) we get

$$ds^2 = dr^2 + r^2 d\varphi^2$$

$$\therefore ds = \sqrt{(dr^2 + r^2 d\varphi^2)}$$

$$\therefore s = \int\sqrt{(dr^2 + r^2 d\varphi^2)} + const.,$$

the value of the constant being such as to make the complete integral vanish at the point whence the arc is estimated.

109. *To find the perpendicular from the origin on the tangent at any given point.*

The equation of the tangent at any given point $x'y'$, (82,) is

$$dy'(x - x') - dx'(y - y') = o,$$

$$\text{or, } dy'.x - dx'.y - (x'dy' - y'dx') = o.$$

Let p be the required perpendicular, and, as in article 84, we shall hence have

$$p = \frac{x'\,dy' - y'\,dx'}{\sqrt{(dy'^2 - dx'^2)}} = \pm \frac{x'\,dy' - y'\,dx'}{ds'}.$$

This reduced for the polar equation, (105,) gives

$$p = \frac{r^2 d\varphi}{\sqrt{(dr^2 + r^2 d\varphi^2)}} = \frac{r^2 d\varphi}{ds}.$$

110. *To find the sectoreal area contained between the curve and any two radii vectores.*

Let us imagine two radii vectores infinitely near to each other, containing the indefinitely small angle $d\varphi$, and subtending the element ds of the curve. The sectoreal element thus formed by these radii vectores and ds may obviously be considered as a plane triangle; and the perpendicular from the origin on the opposite side ds will obviously correspond with that on the tangent. Therefore, p denoting this perpendicular, we shall hence have the area of this sectoral element $= \frac{pds}{2}$.

That is,

$$d.\,(\textit{Sectoreal Area}) = \frac{pds}{2}.$$

But, (109,)

$$p = \frac{xdy - ydx}{ds} = \frac{r^2 d\varphi}{ds}$$

$$\therefore\ d.\,(\textit{Sectoreal Area}) = \frac{xdy - ydx}{2} = \frac{r^2 d\varphi}{2}$$

$$\therefore\ \textit{Sect. Area} = \int \frac{xdy - ydx}{2} + \textit{const.} = \int \frac{r^2 d\varphi}{2} + \textit{const.}$$

The value of the constant must be determined from the position of the radius vector from which the area is to be computed.

111. *To express the inclination of the tangent to a curve with the radius vector.*

Let the required angle under the tangent and radius vector be Tr, and p the perpendicular on the tangent T. Then is

$$\sin Tr = \frac{p}{r}$$

$$\therefore \cos Tr = \frac{\sqrt{(r^2 - p^2)}}{r} \therefore \tan Tr = \frac{p}{\sqrt{(r^2 - p^2)}}.$$

Therefore, substituting the value of p, (109,)

$$\sin Tr = \frac{rd\phi}{ds} = \frac{rd\phi}{\sqrt{(dr^2 + r^2d\phi^2)}},$$

$$\cos Tr = \frac{dr}{ds} = \frac{dr}{\sqrt{(dr^2 + r^2d\phi^2)}},$$

$$\tan Tr = \frac{rd\phi}{dr}.$$

Any of these formulæ will serve for the determination of the required angle.

112. *To express the polar subtangent in terms of r and ϕ.*

The polar subtangent is the straight line drawn from the pole perpendicular to the radius vector and terminating in the tangent. Since

$$\tan Tr = \frac{rd\phi}{dr},$$

we have

$$\textit{Subtangent} = r \tan Tr = \frac{r^2d\phi}{dr}.$$

113. *Cor.* Sometimes the equation of a curve is expressed between the radius vector and the perpendicular on the tangent. In this case we may make use of

$$\tan Tr = \frac{p}{\sqrt{(r^2 - p^2)}},$$

which gives

$$\textit{Subtangent} = \frac{rp}{\sqrt{(r^2 - p^2)}}.$$

114. *To express the radius of curvature of a curve in terms of the radius vector and the polar angle.*

We have, (92,)

$$\varrho = \pm \frac{ds^3}{dy\, d^2x - dx\, d^2y}.$$

Now, pursuing the suppositions used in article 105, we get

$$dy\, d^2x - dx\, d^2y =$$

$$dr\,(sin\,\varphi\, d^2x - cos\,\varphi\, d^2y) + rd\varphi\,(cos\,\varphi\, d^2x + sin\,\varphi\, d^2y)$$

$$= - dr\,(2\, dr\, d\varphi + rd^2\varphi) + rd\varphi\,(d^2r - rd\varphi^2)$$

$$= - d\varphi\,(r^2d\varphi^2 + 2\, dr^2 - rd^2r) - rdr\, d^2\varphi.$$

Hence

$$\varrho = \pm \frac{ds^3}{d\varphi\,(r^2d\varphi^2 + 2\, dr^2 - rd^2r) + rdrd^2\varphi}.$$

By supposing φ to vary independently

$$\varrho = \pm \frac{ds^3}{d\varphi(r^2d\varphi^2 + 2dr^2 - rd^2r)} = \pm \frac{(dr^2 + r^2d\varphi^2)^{\frac{3}{2}}}{d\varphi\,(r^2d\varphi^2 + 2\, dr^2 - rd^2r)}$$

115. *The equation of a curve between the radius vector and the corresponding perpendicular on the tangent being given, to find the angle contained by any two radii vectores.*

By (109),

$$p = \frac{r^2d\varphi}{\sqrt{(dr^2 + r^2d\varphi^2)}}.$$

This solved for $d\varphi$ gives

$$d\varphi = \frac{p\, dr}{r\sqrt{(r^2 - p^2)}},$$

the integral of which between the proposed limits will give the angle sought.

116. *Given the equation between the radius vector and the perpendicular on the tangent, $F\,(r,p) = o$, to find the length of any arc of the curve, and also the sectoreal area between any two radii vectores.*

It has been found, (115), that

$$d\varphi = \frac{p\, dr}{r\sqrt{(r^2 - p^2)}}.$$

This substituted for $d\varphi$ in (108) and (110), we find

$$ds = \frac{rdr}{\surd\,(r^2-p^2)},\quad s = \int \frac{rdr}{\surd\,(r^2-p^2)} + const.;$$

and the *Sectoreal Area* $= \int \frac{prdr}{2\surd\,(r^2-p^2)} + const.$

117. *To express the radius of curvature in terms of the radius vector and the perpendicular on the tangent.*

By (109),

$$p = \frac{r^2 d\varphi}{\surd\,(dr^2 + r^2 d\varphi^2)}.$$

Let $d\varphi$ be supposed constant and

$$\therefore\ dp = \frac{rdr\,d\varphi\,(r^2d\varphi^2 + 2\,dr^2 - rd^2r)}{(dr^2 + r^2d\varphi^2)^{\frac{3}{2}}}.$$

But, (114,)

$$\varrho = \pm \frac{(dr^2 + r^2d\varphi^2)^{\frac{3}{2}}}{d\varphi\,(r^2d\varphi^2 + 2\,dr^2 - rd^2r)}$$

$$\therefore\ \varrho\,dp = \pm\,rdp$$

$$\therefore\ \varrho = \pm\,\frac{rdr}{dp}.$$

118. *The equation of a curve between the radius vector and the perpendicular on the tangent being given, to find the similar equation for its evolute.*

The radius of curvature of the curve at any point rp coincides with the normal and touches the evolute, (96). Let R, P be the radius vector and the perpendicular on the tangent which belong to the evolute at the point of contact. By drawing the figure it will be at once perceived that p and P constitute a rectangle with the tangent and normal of the curve; also that

$$P^2 = r^2 - p^2,$$
$$R^2 = (\varrho - p)^2 + P^2,$$

which inverted, and the value of P by the former substituted in the latter, we get

$$r^2-p^2=P^2,$$

$$(\varrho-p)^2+r^2-p^2=\varrho^2-2p\varrho+r^2=R^2.$$

The value of ϱ being previously determined, (117,) we can by means of these and the given equation of the curve, $f(r,p)=o$, eliminate r and p, which will produce the equation wanted.

119. Let R' and P' be the radius vector and the perpendicular on the tangent which belong to an *involute* of the curve; as the curve is its evolute we have from the foregoing equations, substituting $\frac{R'dR'}{dP'}$ for ϱ,

$$R'^2-P'^2=p^2,$$

$$\left(R'\frac{dR'}{dP'}-P'\right)^2+R'^2-P'^2=r^2.$$

The values of pr by these equations substituted in the equation of the curve we shall find an equation involving $R'P'$ and their differentials. If it can be integrated the equation of the involutes of the curve will thence be found.

END OF PART FIRST.

CORRECTIONS.

At the end of the Preface for "*afford*" read "*to afford.*"
Art. 15, line 2, for "*axes of* x" read "*axis of* x."
Page 14, line 7, for "*substituting*" read "*substitute.*"
Art. 40, line 15, for $+$ read $-$
Art. 45, line 13, for $A'Oa$ read $A'O'a$.
Page 48, first line, for $-$ read $+$